Two-spinor Formulation of First Order Gravity coupled to Dirac Fields

by

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Abstract. *Two-spinor formalism for Einstein Lagrangian is developed. The gravitational field is regarded as a composite object derived from soldering forms. Our formalism is geometrically and globally well-defined and may be used in virtually any 4m-dimensional manifold with arbitrary signature as well as without any stringent topological requirement on space-time, such as parallelizability. Interactions and feedbacks between gravity and spinor fields are considered. As is well known, the Hilbert-Einstein Lagrangian is second order also when expressed in terms of soldering forms. A covariant splitting is then analysed leading to a first order Lagrangian which is recognized to play a fundamental role in the theory of conserved quantities. The splitting and thence the first order Lagrangian depend on a reference spin connection which is physically interpreted as setting the zero level for conserved quantities. A complete and detailed treatment of conserved quantities is then presented.*

Introduction

In the last decade many efforts have been produced in the literature to provide a better understanding of the new geometrodynamical variables proposed by Ashtekar [1, 2]. As it is known, Ashtekar's is a new set of variables for gravity involving soldering forms and connections. The aim of this paper is to present, by using only soldering forms as the independent field variables, a covariant and global first order spinorial splitting of Hilbert's Lagrangian.

A similar splitting was introduced in 1916 by Einstein [3] in order to deal with the problem of the energy of the gravitational field and, more generally, with the

problem of conserved quantities associated to the gravitational field itself. However, Einstein's original splitting was non-covariant and the conserved quantities so-defined were non-covariant as well. Later, it was recognized, originally by Rosen in [4] (see also [5, 6, 7, 8]), that a covariant splitting was possible, provided that a background connection is introduced, which then enters the expression of conserved quantities.

Since it is generally accepted that in General Relativity no absolute quantity should depend on *unphysical background* fields, one is forced to interpret these conserved quantities as *conserved quantities relative to the background* (better, *reference*) *configuration*. On the other hand, in the literature (see [9, 10]) it is well accepted that in General Relativity only relative conserved quantities make sense. This is intuitively clear if one bears in mind that conserved quantities are non-local quantities and that solutions in General Relativity may be globally very different from each other also from a topological viewpoint. Then it sounds reasonable that, e.g., an infinite amount of energy has to be spent to deform a solution so much that its global properties change. In this way, the set of solutions of General Relativity is disconnected into classes, which are physically separated by an infinite potential barrier.

The starting point of this paper is the observation that the Hilbert Lagrangian, expressed in spinorial variables, admits a background-dependent global and covariant splitting, in which the first term is a global formal divergence playing no role at all for the field equations (since divergences have vanishing variational derivatives) and the second term gives a family of first order global Lagrangians, which generate Einstein's field equations. The background field, which parametrizes the new family of global Lagrangians, is a non-dynamical $SL(2,\mathbb{C})$ spin connection. Clearly, the globality of the Lagrangian is useless to ensure the globality of solutions (general covariance of the equations ensures it), but plays a fundamental role in the theory of conserved quantities.

Our formalism has been worked out to deal with interactions between gravity and spinors in a framework which recalls gauge theories in their geometrical formulation, where one starts from a principal fibre bundle over space-time, the so-called *structure bundle* Σ. The structure bundle encodes the symmetry structure of the theory. The *configuration bundle* B is then associated to the structure bundle: i.e., the principal automorphisms of the structure bundle are represented on B by means

of a natural (functorial) action.

In our formalism, gravity is described by the Ashtekar soldering forms, which for the first time are here presented as global sections of a bundle Σ_χ associated to the structure bundle Σ. Globality of soldering forms was already achieved in particular cases (e.g., on parallelizable manifolds), usually at the cost of requiring very stringent topological properties on space-time. Our framework applies to a very wide class of manifolds (namely, to *any* spin manifold).

The bundle Σ_χ has been here called the *bundle of (co)spin-vierbeins* and, as stated above, is built out of Σ in a canonical (functorial) fashion. These spinorial variables are suitably related to spin structures on space-time and any co(spin)-vierbein induces a metric, which is then regarded as a composite object.

In our framework, one does not have to fix the metric $\boldsymbol{g}$ on space-time, give the Lagrangian and thence the field equations (of which $\boldsymbol{g}$ has to be a solution) before defining any spin structure—as on the contrary it is a standard procedure in the literature when dealing with spinors and gravity. Clearly, the standard approach makes sense only when the gravitational field is considered unaffected by spinors, whereas our formalism is able to describe the complete interaction and feedback between gravity and spinor fields.

Thus, a field theory for sections of Σ_χ is considered. A background $SL(2,\mathbb{C})$ spin connection, possibly determined by a background (co)spin-vierbein, is introduced merely in order to globalize (in spinorial variables) the local and non-covariant first order Lagrangian originally given by Einstein, playing no other role but setting the "zero level" for conserved quantities.

1. Spin Structures, spin-frames and soldering forms

Let M be a (real) 4-dimensional orientable manifold which admits a smooth metric $\boldsymbol{g}$ of signature $(+,-,-,-)$ and components $(g_{\mu\nu})$; i.e., we shall assume throughout the sequel that M satisfies the topological requirements which ensure the existence on it of a Lorentzian structure $(M,\boldsymbol{g})$. We also stress that we are *not* fixing $\boldsymbol{g}$, but it is to be understood as determined by the spinorial variables $(e^{AB'}{}_\mu)$ giving the soldering form as defined below and which will be called "(co)spin-vierbeins".

With this end in view, we shall also assume that our space-time M admits a "free spin structure" (see [11, 12, 13, 14] and references therein); i.e., we shall assume the existence of at least one principal fibre bundle Σ over M with structure group

$SL(2,\mathbb{C})$, called the *spin structure bundle*, and at least one strong (i.e. covering the identity map) equivariant morphism $\Lambda : \Sigma \to \mathbb{L}(M)$, $\mathbb{L}(M)$ denoting the principal bundle of linear frames on M. Equivalently, we have the following commutative diagrams

$$\begin{tikzcd} \Sigma \arrow[r, "\Lambda"] \arrow[d] & \mathbb{L}(M) \arrow[d] \\ M \arrow[r, "id_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} \Sigma \arrow[r, "R_S"] \arrow[d, "\Lambda"'] & \Sigma \arrow[d, "\Lambda"] \\ \mathbb{L}(M) \arrow[r, "R_{\hat{l}(S)}"'] & \mathbb{L}(M) \end{tikzcd} \tag{1.1}$$

where $\hat{l} := i \circ l$ is the composed morphism of l, the epimorphism which exhibits $SL(2,\mathbb{C})$ as a two-fold covering of the proper orthochronous Lorentz group $SO(1,3)_0$, with the canonical injection $i\colon SO(1,3)_0 \to GL(4)$ of Lie groups, and R denotes each of the canonical right actions (see [14]).

We call the bundle map Λ a *spin-frame* on Σ and the pair (Σ, Λ) a *free spin structure*.

This definition of spin structure induces metrics on M. In fact, given a spin-frame $\Lambda : \Sigma \to \mathbb{L}(M)$, we can define a metric via the reduced subbundle $SO_0(M, \boldsymbol{g}_\Lambda) \equiv \mathrm{Im}(\Lambda)$ of $\mathbb{L}(M)$. In other words, $\boldsymbol{g}_\Lambda$ is the only *dynamic* metric such that frames in $\mathrm{Im}(\Lambda) \subset \mathbb{L}(M)$ are $\boldsymbol{g}_\Lambda$-orthonormal frames. It is important here to stress that in our picture the metric $\boldsymbol{g}_\Lambda$ is built up *a posteriori*, after a spin-frame has been determined by the field equations in a way which is compatible with the (free) spin structure one has used to define spinors.

This definition of (free) spin structure without fixing any background metric, which already appeared in an original work by van den Heuvel [15], is given with respect to a fixed spin bundle Σ, but permitting variation of spin-frames. The variation of spin-frames induces a variation of the metric. In fact, it has now been established [14] that there is a bijection between spin-frames and sections of a gauge-natural bundle, here denoted by Σ_ρ, a fibre bundle the sections of which represent spin-frames. Such a bundle is given as follows.

Remind that $SL(2,\mathbb{C}) \cong Spin(1,3)_0$ and consider the following left action of the group $GL(4) \times SL(2,\mathbb{C})$ on the manifold $GL(4) \equiv GL(4,\mathbb{R})$

$$\begin{cases} \rho\colon (GL(4) \times SL(2,\mathbb{C})) \times GL(4) \to GL(4) \\ \rho\colon ((A^\mu{}_\nu, t^A{}_B), e^a{}_\mu) \mapsto (\Lambda^a{}_b(\boldsymbol{t}) e^b{}_\nu (\boldsymbol{A}^{-1})^\nu{}_\mu) \end{cases} \tag{1.2}$$

together with the associated bundle $\Sigma_\rho := W^{1,0}(\Sigma) \times_\rho GL(4)$, where $W^{1,0}(\Sigma) := \mathbb{L}(M) \times_M \Sigma$ denotes the principal prolongation of order $(1,0)$ of the principal fibre bundle Σ and $\times_M$ denotes the fibred product of two bundles over the same base manifold. The bundle $\mathbb{L}(M) \times_M \Sigma$ is a principal fibre bundle with structure group $GL(4) \times SL(2,\mathbb{C})$. It turns out that Σ_ρ is a fibre bundle associated to $W^{1,0}(\Sigma)$, i.e. a gauge-natural bundle of order $(1,0)$. The bundle Σ_ρ has been called the *bundle of (co)spin-tetrads* [14].

Under these assumptions, to each point $p \in M$ we can assign a complex 2-dimensional vector space $S_p(M)$ equipped with a non-degenerate symplectic form (2-form). (The components of) a generic element of $S_p(M)$ will be denoted by ξ^A and (the components of) the corresponding symplectic form by ε_{AB} (its inverse will be denoted by ε^{AB} and is such that $\varepsilon_{AC}\varepsilon^{AB} = \delta_C{}^B$). The complex conjugate vector space associated with $S_p(M)$ will be denoted by $\overline{S_p(M)}$, its elements by $\bar{\xi}^{A'}$ and the symplectic form by $\varepsilon_{A'B'}$. Since the group preserving the structure on $S_p(M)$ is $SL(2,\mathbb{C})$, ξ^A will be called a $SL(2,\mathbb{C})$ spinor at $p \in M$ or, for short, a *two-spinor*. Equivalently, two-spinors may be defined via the standard linear action of $SL(2,\mathbb{C})$ on $\mathbb{C}^2$ and we shall denote by $S(M) := (\Sigma \times \mathbb{C}^2)/SL(2,\mathbb{C})$ the vector bundle associated to the principal fibre bundle Σ by means of this action. The spin connection can then be used to construct a $SL(2,\mathbb{C})$ covariant derivative of spinor fields.

Now, if we wish to consider a field theory in which spinorial variables are dynamical, we must first construct a fibre bundle the sections of which represent spin-vierbeins. To this end, we need to make a short digression on complex structures in order to clarify our notation. The material presented here is standard.

Recall that, if E is a complex vector space, then its *conjugate space* $\bar{E}$ is obtained from E by redefining scalar multiplication. The new scalar multiplication by $m \in \mathbb{C}$ is the old scalar multiplication by $\bar{m}$. The axioms of a complex vector space are easily seen to be satisfied on $\bar{E}$. Usually, one agrees to denote by $\bar{\boldsymbol{v}}$ the vector $\boldsymbol{v}$ when it is considered as an element of $\bar{E}$. If $\boldsymbol{f}\colon E \to F$ is a linear map of complex vector spaces, then one defines a linear map $\bar{\boldsymbol{f}}\colon \bar{E} \to \bar{F}$ by $\bar{\boldsymbol{f}}(\bar{\boldsymbol{v}}) := \overline{\boldsymbol{f}(\boldsymbol{v})}$. For any complex vector space E the spaces $(\bar{E})^* := \{\,\boldsymbol{\alpha}\colon \bar{E} \to \mathbb{C} \mid \boldsymbol{\alpha} \text{ is linear}\,\}$ and $\overline{(E^*)} := \{\,\bar{\boldsymbol{\beta}} \mid \boldsymbol{\beta}\colon E \to \mathbb{C} \text{ is linear}\,\}$ are naturally isomorphic. The isomorphism $\boldsymbol{\iota}\colon (\bar{E})^* \to \overline{(E^*)}$ is given by $\boldsymbol{\iota}(\boldsymbol{\alpha}) := \bar{\boldsymbol{\beta}}$, where $\langle \boldsymbol{\beta}, \boldsymbol{v}\rangle = \overline{\langle \boldsymbol{\alpha}, \bar{\boldsymbol{v}}\rangle}$ and $\boldsymbol{v} \in E$. Owing to such an isomorphism, we shall identify the space $(\bar{E})^*$ with $\overline{(E^*)}$ and denote it $\bar{E}^*$. Let us also recall that, in general, for a complex vector space E there is no

canonical way to represent E as the direct sum of two real spaces, the *real* and *imaginary* parts of E, although each complex vector space E admits a *real form* obtained by taking the same set and restricting the scalars to be real. An additional *real structure* in E (see, e.g., [16]) is a linear map $C: E \to \bar{E}$ such that $\bar{C}C = \mathrm{id}_E$. Any vector $\boldsymbol{v} \in E$ splits as $\boldsymbol{v} = \boldsymbol{v}^+ + \boldsymbol{v}^-$, where we set $\boldsymbol{v}^\pm := \frac{1}{2}(\boldsymbol{v} \pm \bar{C}\bar{\boldsymbol{v}})$. We have a direct sum decomposition of E into two real vector spaces E^+ and E^- such that $\boldsymbol{v} \in E^\pm$ iff $\bar{\boldsymbol{v}} = \pm C\boldsymbol{v}$. On the vector space $E = \mathbb{C}^2 \otimes \bar{\mathbb{C}}^2$ consider the real structure $C: \mathbb{C}^2 \otimes \bar{\mathbb{C}}^2 \to \bar{\mathbb{C}}^2 \otimes \mathbb{C}^2$ defined by $C(\boldsymbol{u} \otimes \bar{\boldsymbol{v}}) := \bar{\boldsymbol{v}} \otimes \boldsymbol{u}$. The real space E^+ is the real space of Hermitian tensors spanned by elements of the form $\boldsymbol{u} \otimes \bar{\boldsymbol{v}}$. A generic element of E^+ is written as $\boldsymbol{\phi} = \phi^{AB'}\boldsymbol{c}_A \otimes \boldsymbol{c}_{B'}$ where $\overline{\phi^{AB'}} = \phi^{BA'}$ and $(\boldsymbol{c}_{A'})$ is the basis of $\bar{\mathbb{C}}^2$ consisting of the same vectors as $(\boldsymbol{c}_A)$. Hermitian tensors of the real vector space E^+ are also called *real* (see Ref. [17]).

Now, let V be the open subset of $E^+ \otimes (\mathbb{R}^4)^*$ consisting of all invertible real linear maps $\boldsymbol{\phi}: \mathbb{R}^4 \to E^+$. An element $\boldsymbol{\phi} \equiv \phi^{AB'}{}_\mu \boldsymbol{c}_A \otimes \boldsymbol{c}_{B'} \otimes \boldsymbol{c}^\mu$ of the vector space $\mathbb{C}^2 \otimes \bar{\mathbb{C}}^2 \otimes (\mathbb{R}^4)^*$ belongs to V iff the following conditions hold

$$\overline{\phi^{AB'}{}_\mu} = \phi^{BA'}{}_\mu, \tag{1.3a}$$

$$\phi^{AB'}{}_\mu \phi_{AB'}{}^\nu = \delta^\nu_\mu, \tag{1.3b}$$

$$\phi^{AB'}{}_\mu \phi_{CD'}{}^\mu = \delta^A_C \delta^{B'}_{D'}, \tag{1.3c}$$

where $(\phi_{AB'}{}^\mu)$ denote the components of the inverse element $\boldsymbol{\phi}^{-1} \equiv \phi_{AB'}{}^\mu \boldsymbol{c}^A \otimes \boldsymbol{c}^{B'} \otimes \boldsymbol{c}_\mu$; here indices are *not* raised or lowered with $g_{\mu\nu}$ or ε_{AB}, although, if we define $g_{\mu\nu} := \phi^{AB'}{}_\mu \phi^{CD'}{}_\nu \varepsilon_{AC}\varepsilon_{B'D'}$, the n we find $\phi_{BA'}{}^\mu = \phi_{\dot{B}\dot{A'}}{}^{\cdot\cdot\mu}$, where on the r.h.s. the tensor index μ is raised using $g_{\mu\nu}$ and the indices AB' are lowered using ε_{AB} and $\varepsilon_{A'B'}$, respectively. Formulae (1.3b) and (1.3c) reflect the fact that the composed linear map $\boldsymbol{\phi}^{-1} \circ \boldsymbol{\phi}$ is the identity map on $\mathbb{R}^4$ and $\boldsymbol{\phi} \circ \boldsymbol{\phi}^{-1}$ is the identity map on E^+.

We are at last in a position to consider the following left action on V

$$\begin{cases} \chi: (GL(4) \times SL(2,\mathbb{C})) \times V \to V \\ \chi: ((A^\mu{}_\nu, t^A{}_B), W^{AB'}{}_\nu) \mapsto (t^A{}_C t^{B'}{}_{D'} W^{CD'}{}_\nu (\boldsymbol{A}^{-1})^\nu{}_\mu) \end{cases} \tag{1.4}$$

together with the associated bundle $\Sigma_\chi := (\mathbb{L}(M) \times_M \Sigma) \times_\chi V$. According to the theory of gauge-natural bundles and gauge-natural operators (see Ref. [18]), Σ_χ turns out to be a fibre bundle associated to $W^{1,0}(\Sigma)$, i.e. a gauge-natural bundle

of order $(1,0)$. Local coordinates on the bundle Σ_χ will be denoted by $(x^\mu, e^{AB'}{}_\mu)$. A section of Σ_χ will be called a *(co)spin-vierbein*. Equivalently, a (co)spin-vierbein may be regarded as an Ashtekar soldering form, i.e. as an (invertible) linear map $\boldsymbol{A}_p\colon T_pM \to S_p(M)\otimes\overline{S_p(M)}$ at each point $p \in M$, with the property of being "real", i.e. such that the components $(A^{AB'}{}_\mu)$ of $\boldsymbol{A}_p$ constitute a Hermitian matrix for each value of μ.

It is possible to construct another bundle Σ_τ with the same fibre V by considering the following left action on the $SL(2,\mathbb{C})$-manifold V:

$$\begin{cases} \tau\colon SL(2,\mathbb{C}) \times V \to V \\ \tau\colon (t^A{}_B, F^{AB'}{}_a) \mapsto (t^A{}_C t^{B'}{}_{D'} F^{CD'}{}_b \Lambda^b{}_a(\boldsymbol{t}^{-1})) \end{cases}. \tag{1.5}$$

The bundle $\Sigma_\tau := \Sigma \times_\tau V$ is a fibre bundle associated to the principal fibre bundle Σ, also denoted by $W^0(\Sigma)$, with structure group $SL(2,\mathbb{C})$. It turns out that Σ_τ is a gauge-natural bundle of order zero, i.e. associated to the "trivial" (zeroth order) principal prolongation of Σ. Local coordinates on the bundle Σ_τ will be denoted by $(x^\mu, M^{AB'}{}_a)$. A special choice for Σ_τ is the section $\sigma_{\mathrm{IW}}\colon M \to \Sigma_\tau$ whose components, in any system of local coordinates, are given by the "Infeld-van der Waerden symbols" [11, 19–22], i.e. the section

$$\sigma_{\mathrm{IW}}\colon (x^\alpha) \mapsto (x^\alpha, M^{AB'}{}_a = \sigma^{AB'}{}_a). \tag{1.6}$$

The *Infeld-van der Waerden section* σ_{IW} shall be called the *canonical section* of Σ_τ. It is a global section because its components, i.e. the Infeld-van der Waerden symbols, are the components (in the standard fibre) of an $SL(2,\mathbb{C})$-invariant tensor.

In fact, whenever one has a principal fibre bundle (P, M, G, π) with structure group G and a left action of G on some real or complex vector space V, it is possible, if we are given an invariant vector of V with respect to G, i.e. if we suppose there exists a vector $\boldsymbol{v} \in V$ such that $g \cdot \boldsymbol{v} = \boldsymbol{v}$ for all $g \in G$, to construct (using the transition functions of P) a global section s of the associated bundle $(P \times V)/G$, whose components are the components of $\boldsymbol{v} \in V$ with respect to a basis chosen in V.

The *canonical* Infeld-van der Waerden section induces a *canonical isomorphism* (over the identity) of real fibre bundles, locally represented by:

$$\begin{cases} \Phi_{\mathrm{IW}}\colon \Sigma_\chi \to \Sigma_\rho \\ \Phi_{\mathrm{IW}}\colon (x^\alpha, e^{AB'}{}_\mu) \mapsto (x^\alpha, e^a{}_\mu = \sigma_{AB'}{}^a e^{AB'}{}_\mu) \end{cases}. \tag{1.7}$$

We are in a position to state the following (cf. Ref. [14])

Proposition. *There is a bijection between spin-frames and the sections of the gauge-natural bundle Σ_χ, i.e. between spin-frames and (co)spin-vierbeins (Ashtekar soldering forms).*

In other words, the above proposition asserts that we may represent spin-frames with *dynamical* (global) Ashtekar soldering forms, and this fact is crucial if we want to consider a field theory in which spin-frames are dynamical.

2. Standard General Relativity in two-spinor formalism

In our theory the standard "Hilbert" spinor Lagrangian is built out of the soldering form variables, or our "(co)spin-vierbeins". Of course, it turns out to be a second order Lagrangian theory in these variables.

In fact, define

$$\boldsymbol{\Sigma}^{AB} := \frac{i}{2}\varepsilon_{A'B'}\,\boldsymbol{\vartheta}^{AA'} \wedge \boldsymbol{\vartheta}^{BB'}, \tag{2.1}$$

$\boldsymbol{\vartheta}^{AA'} = e^{AA'}{}_{\mu}\,\mathrm{d}x^\mu$ being the *Ashtekar soldering form* [1, 2, 23]. Define also

$$\boldsymbol{\Omega}^A{}_B := \mathrm{d}_{\mathrm{H}}\boldsymbol{\omega}^A{}_B + \boldsymbol{\omega}^A{}_C \wedge \boldsymbol{\omega}^C{}_B,$$

where d_{H} is the horizontal differential [24] and the coefficients of the (unprimed) spin connection $\boldsymbol{\omega}^A{}_B \equiv \omega^A{}_{B\mu}\,\mathrm{d}x^\mu$ are regarded as being uniquely determined by the spin-vierbeins and their first partial derivatives $(e^{AB'}{}_{\mu\nu})$ via the relation (cf. [25, 26])

$$\omega^A{}_{B\mu} = \frac{1}{2}(e_{BA'}{}^{\nu}e^{AA'}{}_{[\mu\nu]} + e^{AA'\rho}e_{CC'\mu}e_{BA'}{}^{\nu}e^{CC'}{}_{[\rho\nu]} + e^{AA'\nu}e_{BA'[\nu\mu]}). \tag{2.2}$$

Since we aim to describe a spinor field (without any further gauge symmetry) in interaction with gravity, our *configuration space* will be assumed to be the following bundle

$$B = \Sigma_\chi \times_M \Sigma_\gamma, \tag{2.3}$$

where $\Sigma_\gamma := \Sigma \times_\gamma E$ is the vector bundle associated to the principal bundle Σ via the obvious representation γ of $SL(2,\mathbb{C})$ on the vector space $E := \bar{\mathbb{C}}^2 \oplus (\mathbb{C}^2)^*$. The bundle Σ_γ is then isomorphic to $\bar{S}(M) \oplus_M S^*(M)$.

Consequently, the Lagrangian will be chosen of the following form:

$$\boldsymbol{\mathcal{L}}\colon J^2\Sigma_\chi \times_M J^1\Sigma_\gamma \to \Lambda^4 T^*M. \tag{2.4}$$

According to the principle of minimal coupling, the Lagrangian $\boldsymbol{\mathcal{L}}$ is assumed to split into two parts $\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}_H + \boldsymbol{\mathcal{L}}_D$, where

$$\begin{cases} \boldsymbol{\mathcal{L}}_H \colon J^2\Sigma_\chi \to \Lambda^4 T^*M \\ \boldsymbol{\mathcal{L}}_H = -\dfrac{1}{\kappa}\boldsymbol{\Omega}_{AB} \wedge \boldsymbol{\Sigma}^{AB} + \text{c.c.} \qquad (\kappa := 8\pi G/c^4) \end{cases} \tag{2.5}$$

is the standard "Hilbert" spinor Lagrangian, "c.c." stands for the complex conjugate of the preceding term and $\boldsymbol{\mathcal{L}}_D \colon J^1(\Sigma_\chi \times_M \Sigma_\gamma) \to \Lambda^4 T^*M$ is the two-spinor equivalent of the Dirac Lagrangian [14, 26]

$$\boldsymbol{L}_D = \left[\frac{i}{2}(\tilde{\boldsymbol{\Psi}} \cdot \gamma^a \cdot \nabla_a \boldsymbol{\Psi} - \widetilde{\nabla_a \boldsymbol{\Psi}} \cdot \gamma^a \cdot \boldsymbol{\Psi}) - m\tilde{\boldsymbol{\Psi}} \cdot \boldsymbol{\Psi}\right]\boldsymbol{\Sigma},$$

where $\tilde{\boldsymbol{\Psi}} := \boldsymbol{\Psi}^\dagger \cdot \gamma_0$ is called the *Dirac adjoint* of $\boldsymbol{\Psi}$, $\gamma^a := \eta^{ab}\gamma_b$, the dot '$\cdot$' denotes matrix product and $\boldsymbol{\Sigma} := {}^4e\,\mathbf{ds}$ is the standard 4-form, ${}^4e$ being the determinant of $(e^a{}_\mu)$ (or, equivalently, the determinant of $(e^{AB'}{}_\mu)$) and $\mathbf{ds} := \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ the (local) volume element. If we set $\boldsymbol{\Psi} =: \boldsymbol{\psi} \oplus \boldsymbol{\varphi}$, $\tilde{\boldsymbol{\Psi}} =: \bar{\boldsymbol{\varphi}} \oplus \bar{\boldsymbol{\psi}}$ (see Ref. [17]) and define for any vector field $\boldsymbol{v}$ on M

$$\not{v}\boldsymbol{\Psi} \equiv v^a\gamma_a \cdot \boldsymbol{\Psi} := \sqrt{2}(v^{AA'}\varphi_A \boldsymbol{f}_{A'} \oplus v_{AA'}\psi^{A'}\boldsymbol{f}^A)$$

(which implies our *Clifford product* has the form $\gamma_a\cdot\gamma_b + \gamma_b\cdot\gamma_a = \eta_{ab}\mathbb{I}_4$), $\boldsymbol{\psi} \equiv \psi^{A'}\boldsymbol{f}_{A'}$ being a section of $\bar{S}(M)$ and $\boldsymbol{\varphi} \equiv \varphi_A \boldsymbol{f}^A$ a section of $S^*(M)$, it is straightforward to prove that $\boldsymbol{\mathcal{L}}_D$ has the following expression

$$\boldsymbol{\mathcal{L}}_D = \left\{\left[\frac{i\sqrt{2}}{2}(\bar{\varphi}_{A'}\nabla^{AA'}\varphi_A + \bar{\psi}^A\nabla_{AA'}\psi^{A'}) - m\,\varphi_A\bar{\psi}^A\right] + \text{c.c.}\right\}\boldsymbol{\Sigma}, \tag{2.6}$$

where $\nabla_{AA'} := e_{AA'}{}^\mu\nabla_\mu$.

Notice that in this formalism Dirac's equation $(i\not{\nabla} - m)\boldsymbol{\Psi} = \mathbf{0}$ takes the symmetric form

$$\begin{cases} i\sqrt{2}\nabla_{AA'}\psi^{A'} - m\varphi_A = 0 \\ i\sqrt{2}\nabla^{AA'}\varphi_A - m\psi^{A'} = 0 \end{cases}. \tag{2.7}$$

Thus the total Lagrangian $\boldsymbol{\mathcal{L}}$ can be simply represented in terms of the variables discussed above together with their partial derivatives up to the second order included $(e^{AB'}{}_\mu, e^{AB'}{}_{\mu\nu}, e^{AB'}{}_{\mu\nu\rho})$.

According to recent results [27], to each higher order Lagrangian there corresponds at least one global *Poincaré-Cartan form*. Such a form is unique for first

order theories; in the second order case uniqueness is lost, although there is still a canonical choice, which we will now describe. Let

$$\boldsymbol{L} \equiv L(x^\alpha; y^{\boldsymbol{a}}, y^{\boldsymbol{a}}{}_\lambda, y^{\boldsymbol{a}}{}_{\lambda\mu})\, \mathbf{ds}$$

be a second order Lagrangian, where $y^{\boldsymbol{a}}$ is a field of arbitrary nature. Define the *momenta* by setting

$$f_{\boldsymbol{a}}{}^{\lambda\mu} := \frac{\partial L}{\partial y^{\boldsymbol{a}}{}_{\lambda\mu}}, \qquad f_{\boldsymbol{a}}{}^{\lambda} := \frac{\partial L}{\partial y^{\boldsymbol{a}}{}_{\lambda}} - \mathrm{d}_\mu \frac{\partial L}{\partial y^{\boldsymbol{a}}{}_{\lambda\mu}},$$

where d_μ denotes the formal derivative [14, 24]. The Poincaré-Cartan form associated to $\boldsymbol{L}$ is thence given by

$$\boldsymbol{\Theta}(\boldsymbol{L}) := \boldsymbol{L} + (f_{\boldsymbol{a}}{}^{\lambda}\, \mathrm{d}_{\mathrm{V}} y^{\boldsymbol{a}} + f_{\boldsymbol{a}}{}^{\lambda\mu}\, \mathrm{d}_{\mathrm{V}} y^{\boldsymbol{a}}{}_\mu) \wedge \mathbf{ds}_\lambda, \tag{2.8}$$

where d_{V} is the vertical differential [24] and we set $\mathbf{ds}_\lambda := \partial_\lambda \rfloor \mathbf{ds}$, '$\rfloor$' denoting inner product.

The knowledge of the Poincaré-Cartan form enables us to calculate the so-called *energy density flow* of the Lagrangian in question. In fact, if $\boldsymbol{L}$ is a Lagrangian defined on the k-th order prolongation of a gauge-natural bundle B (see Ref. [18]) and $\boldsymbol{\Xi}$ is the generator of a one-parameter subgroup of automorphisms of B, the energy density flow associated to $\boldsymbol{L}$ along the vector field $\boldsymbol{\Xi}$ is given by (cf. [28, 8])

$$\boldsymbol{E}(\boldsymbol{L}, \boldsymbol{\Xi}) \equiv E^\alpha(\boldsymbol{L}, \boldsymbol{\Xi})\, \mathbf{ds}_\alpha := -\operatorname{Hor}[\tilde{\boldsymbol{\Xi}} \rfloor \boldsymbol{\Theta}(\boldsymbol{L})],$$

where Hor denotes the horizontal operator on forms [24] and $\tilde{\boldsymbol{\Xi}}$ is the $(2k-1)$-th order prolongation of $\boldsymbol{\Xi}$ (we stress that the word "energy" is used here in the broader sense of "conserved Nöther current"). In particular, for a second order Lagrangian one finds:

$$\boldsymbol{E}(\boldsymbol{L}, \boldsymbol{\Xi}) = (f_{\boldsymbol{a}}{}^{\alpha}\, \pounds_{\boldsymbol{\Xi}} y^{\boldsymbol{a}} + f_{\boldsymbol{a}}{}^{\alpha\mu}\, \pounds_{\boldsymbol{\Xi}} y^{\boldsymbol{a}}{}_\mu - L\xi^\alpha)\, \mathbf{ds}_\alpha, \tag{2.9}$$

$\boldsymbol{\xi}$ being the projection of $\boldsymbol{\Xi}$ on M.

Our Poincaré-Cartan form associated to $\boldsymbol{\mathcal{L}}_H$ is

$$\boldsymbol{\Theta}(\boldsymbol{\mathcal{L}}_H) = \boldsymbol{\mathcal{L}}_H - \frac{1}{\kappa}(\boldsymbol{V}_{AB} \wedge \boldsymbol{\Sigma}^{AB} + \text{c.c.}),$$

where $\boldsymbol{V}_{AB} := 1/2\, e_A{}^{A'}{}_\alpha e_{BA'}{}^\beta\, \mathrm{d}_\mathrm{V}\Gamma^\alpha{}_{\beta\mu} \wedge \mathrm{d}x^\mu$ and $\Gamma^\alpha{}_{\beta\mu}$ is the Levi-Civita connection induced by the metric $g_{\mu\nu}$, uniquely and unequivocally determined by the soldering form via the relation

$$g_{\mu\nu} = e^{AB'}{}_\mu e^{CD'}{}_\nu \varepsilon_{AC}\varepsilon_{B'D'}.$$

An easy calculation shows that the expression for $\boldsymbol{E}(\boldsymbol{\mathcal{L}}_H)$ is

$$\boldsymbol{E}(\boldsymbol{\mathcal{L}}_H, \boldsymbol{\Xi}) = -\frac{1}{\kappa} G^\alpha{}_\beta \xi^\beta \boldsymbol{\Sigma}_\alpha + \frac{1}{2\kappa} \mathrm{d}_\mathrm{H}(\nabla_A^{A'} \xi_{BA'} \boldsymbol{\Sigma}^{AB} + \text{c.c.}), \tag{2.10}$$

where $G^\alpha{}_\beta$ is the Einstein tensor, $\boldsymbol{\Sigma}_\alpha := \partial_\alpha \rfloor \boldsymbol{\Sigma}$ and we set $\xi^{AA'} := e^{AA'}{}_\mu \xi^\mu$, (ξ^μ) being the components of $\boldsymbol{\xi}$ in a local chart.

As one can see from (2.10), $\boldsymbol{E}(\boldsymbol{\mathcal{L}}_H, \boldsymbol{\Xi})$ is conserved in vacuum along any solution of the field equations $G_{\mu\nu} = 0$, while of course it is not in interaction with matter. The 2-form

$$\boldsymbol{U}(\boldsymbol{\mathcal{L}}_H, \boldsymbol{\Xi}) := \frac{1}{2\kappa} \nabla_A^{A'} \xi_{BA'} \boldsymbol{\Sigma}^{AB} + \text{c.c.} \tag{2.11}$$

is called the *Hilbert superpotential*: it is straightforward to show that it is nothing but the half of the well known Komar superpotential [29]. Therefore, setting (in spherical coordinates) $\boldsymbol{\xi} = \partial/\partial t$ and integrating (2.11) on a spherical surface, it will yield half the mass for the Schwarzschild solution, but the correct angular momentum for the (Schwarzschild and) Kerr solution (see Ref. [8]).

We can now tackle the spinorial contribution, writing down the Poincaré-Cartan form associated to $\boldsymbol{\mathcal{L}}_D$. Using (2.8), which is of course still valid for first order Lagrangians as a trivial subcase, we find:

$$\begin{aligned}\boldsymbol{\Theta}(\boldsymbol{\mathcal{L}}_D) = \boldsymbol{\mathcal{L}}_D + \Big\{ &\tfrac{i\sqrt{2}}{2} [\bar\varphi_{A'} e^{BA'\alpha}\, \mathrm{d}_\mathrm{V}\varphi_B + \bar\psi^A e_{AB'}{}^\alpha\, \mathrm{d}_\mathrm{V}\psi^{B'} \\ &-\tfrac{1}{2}(\bar\varphi_{A'}\varphi_B e^B{}_{C'}{}^\mu e_C{}^{A'\alpha} + \bar\psi^A \psi^{B'} e_{CB'}{}^\mu e_{AC'}{}^\alpha)\, \mathrm{d}_\mathrm{V} e^{CC'}{}_\mu] + \text{c.c.} \Big\} \wedge \boldsymbol{\Sigma}_\alpha.\end{aligned} \tag{2.12}$$

Resorting as usual to (2.9) and making use of the relation (cf. [14])

$$\pounds_{\boldsymbol{\Xi}} e^{AA'}{}_\mu = \nabla_\mu \xi^\nu e^{AA'}{}_\nu - e^{BA'}{}_\mu \mathrm{V}\Xi^A{}_B - e^{AB'}{}_\mu \overline{\mathrm{V}\Xi}^{A'}{}_{B'}, \tag{2.13}$$

where $\mathrm{V}\Xi^A{}_B$ is the vertical part of $\Xi^A{}_B$ with respect to the dynamical connection $\omega^A{}_{B\mu}$, i.e. $\mathrm{V}\Xi^A{}_B := \Xi^A{}_B - \omega^A{}_{B\mu}\xi^\mu$, $(\xi^\mu, \Xi^A{}_B)$ obviously being the components of $\boldsymbol{\Xi}$ in a local chart, we finally get

$$\boldsymbol{E}(\boldsymbol{\mathcal{L}}_D, \boldsymbol{\Xi}) = T^\alpha{}_\beta \xi^\beta \boldsymbol{\Sigma}_\alpha + \mathrm{d}_\mathrm{H} \boldsymbol{U}(\boldsymbol{\mathcal{L}}_D, \boldsymbol{\Xi}), \tag{2.14}$$

where $T^\alpha{}_\beta$ is the energy-momentum tensor associated to $\boldsymbol{\mathcal{L}}_D$ and we set

$$\boldsymbol{U}(\boldsymbol{\mathcal{L}}_D, \boldsymbol{\Xi}) := \frac{i\sqrt{2}}{4}\xi^{A'}_A(\bar{\varphi}_{A'}\varphi_B - \bar{\psi}_B\psi_{A'})\boldsymbol{\Sigma}^{AB} + \text{c.c.} \tag{2.15}$$

Thus, the *total energy density flow*

$$\boldsymbol{E}(\boldsymbol{\mathcal{L}}, \boldsymbol{\Xi}) \equiv \boldsymbol{E}(\boldsymbol{\mathcal{L}}_H, \boldsymbol{\Xi}) + \boldsymbol{E}(\boldsymbol{\mathcal{L}}_D, \boldsymbol{\Xi})$$

appears to be conserved "on shell" (i.e. along any solution of the field equations), owing to the Einstein equations $G^\alpha{}_\beta = \kappa T^\alpha{}_\beta$ and in accordance with the general theory [28]. As a consequence, the 2-form

$$\boldsymbol{U}(\boldsymbol{\mathcal{L}}, \boldsymbol{\Xi}) \equiv \boldsymbol{U}(\boldsymbol{\mathcal{L}}_H, \boldsymbol{\Xi}) + \boldsymbol{U}(\boldsymbol{\mathcal{L}}_D, \boldsymbol{\Xi})$$

can be uniquely identified as the *(total) superpotential* of the theory. Notice that the vertical contribution, i.e. the one containing $\mathrm{V}\boldsymbol{\Xi}$, vanishes identically off shell in (2.14). So, even though the interpretation of our conserved currents could result difficult in principle as we enlarged the symmetry group by adding the vertical transformations, we find out that actually this is not the case.

3. Global first order spinor Einstein Lagrangians

In the theory we developed, we chose as the gravitational part of our Lagrangian $\boldsymbol{\mathcal{L}}$ the usual "Hilbert" spinor Lagrangian (2.5). Another possible candidate is the background-dependent family of *global first order* Lagrangians

$$\begin{cases} \boldsymbol{\mathcal{L}}_G \colon J^1\Sigma_\chi \to \Lambda^4 T^*M \\ \boldsymbol{\mathcal{L}}_G := -\dfrac{1}{\kappa}(\mathbf{K}_{AB} + \boldsymbol{Q}^C{}_A \wedge \boldsymbol{Q}_{BC}) \wedge \boldsymbol{\Sigma}^{AB} + \text{c.c.} \end{cases}, \tag{3.1}$$

where $\mathbf{K}_{AB}$ is the curvature 2-form of a background spin connection $\boldsymbol{\beta}_{AB}$ (see Ref. [30] for the basic formalism) and we set $\boldsymbol{Q}_{AB} := \boldsymbol{\omega}_{AB} - \boldsymbol{\beta}_{AB}$.
If we take $\boldsymbol{\beta}_{AB} \equiv \mathbf{0}$ in (3.1), we recover but the *local* non-covariant first order *spinor Einstein Lagrangian* of Møller-Nester [31, 32]; see also Refs. [33] and [34].

We shall call the Lagrangian (3.1), in a given background, the *global first order spinor Einstein Lagrangian.*

Again, relying on the pull-back properties of Poincaré-Cartan forms [24], we find

$$\boldsymbol{\Theta}(\boldsymbol{\mathcal{L}}_G) = \boldsymbol{\mathcal{L}}_G + \frac{1}{\kappa}[(\mathrm{d_V}\boldsymbol{\Sigma}^{AB} + \boldsymbol{\Sigma}^{AC} \wedge \boldsymbol{Z}^B{}_C + \boldsymbol{\Sigma}^{BC} \wedge \boldsymbol{Z}^A{}_C) \wedge \boldsymbol{Q}_{AB} + \text{c.c.}], \tag{3.2}$$

where we set $\boldsymbol{Z}^A{}_B := 1/2\, e^{AA'}{}_\mu\, \mathrm{d_V} e_{BA'}{}^\mu$.

Now, using (2.9), we can calculate the energy density flow, which, after some manipulations, appears to be

$$\boldsymbol{E}(\boldsymbol{\mathcal{L}}_G,\Xi) = -\frac{1}{\kappa}G^\alpha{}_\beta\xi^\beta\boldsymbol{\Sigma}_\alpha + \mathrm{d_H}\boldsymbol{U}(\boldsymbol{\mathcal{L}}_G,\Xi) + \frac{1}{2\kappa}(e_B{}^{A'}{}_\alpha e_{AA'}{}^\beta \pounds_{\boldsymbol{\xi}}\mathrm{B}^\alpha{}_{\beta\mu}\,\mathrm{d}x^\mu\wedge\boldsymbol{\Sigma}^{AB} + \text{c.c.}) \tag{3.3}$$

with

$$\boldsymbol{U}(\boldsymbol{\mathcal{L}}_G,\Xi) := \boldsymbol{U}(\boldsymbol{\mathcal{L}}_H,\Xi) + \frac{1}{\kappa}\,\boldsymbol{\xi}\rfloor(\boldsymbol{Q}_{AB}\wedge\boldsymbol{\Sigma}^{AB} + \text{c.c.}). \tag{3.4}$$

Here, as for the background linear connection $\mathrm{B}^\alpha{}_{\beta\mu}$, we have two possible choices: if $\mathrm{B}^A{}_{B\mu}$ is given—*mutatis mutandis*—by formula (2.2) via a background soldering form $f^{AB'}{}_\mu$, then $\mathrm{B}^\alpha{}_{\beta\mu}$ is taken to be the Levi-Civita connection of the "*induced* background metric" $h_{\mu\nu} := f^{AB'}{}_\mu f^{CD'}{}_\nu \varepsilon_{AC}\varepsilon_{B'D'}$; otherwise, i.e. if $\mathrm{B}^A{}_{B\mu}$ is a *generic* background $SL(2,\mathbb{C})$ connection and $f^{AB'}{}_\mu$ a further background soldering form, $\mathrm{B}^\alpha{}_{\beta\mu}$ is taken to be the linear connection (with torsion) given by the following formula:

$$\mathrm{B}^\alpha{}_{\beta\mu} = f_{AB'}{}^\alpha \mathrm{B}^A{}_{C\mu} f^{CB'}{}_\beta + f_{BA'}{}^\alpha \bar{\mathrm{B}}^{A'}{}_{C'\mu} f^{BC'}{}_\beta + f_{AB'}{}^\alpha f^{AB'}{}_{\beta\mu}. \tag{3.5}$$

Let us note that, considering the soldering form $f^{AB'}{}_\mu$ more generally as an object on a $GL(2,\mathbb{C})$-principal bundle P (…they are not spinors!), formula (3.5) gives us a bijection between $GL(2,\mathbb{C})$-principal connections on P and (complex) linear connections on M (i.e. complex linear connections on the complexified tangent bundle $(TM)^\mathbb{C}$ over the real manifold M). These objects are classical and appear in the "old" literature (see Refs. [20, 21, 25, 35]). That is why, in this case, we shall call them the *Infeld-van der Waerden variables.*

Now, comparing (3.3) with (2.10), we see that in (3.3) we have two additional terms containing the Lie derivative of the background connection. So $\boldsymbol{E}(\boldsymbol{\mathcal{L}}_G,\Xi)$ will be conserved on shell only for those vector fields $\boldsymbol{\xi}$ such that $\pounds_{\boldsymbol{\xi}}\mathrm{B}^\alpha{}_{\beta\mu} = 0$, e.g. for Killing vector fields of the background linear connection. It is possible to show [8] that the additional background contribution in (3.4), when integrated on a spherical surface with $\boldsymbol{\xi} = \partial/\partial t$, restores the expected value for the mass of the Schwarzschild solution, if the Levi-Civita connection of the Minkowski metric is chosen as the (obviously) appropriate background. Of course, the angular momentum associated to the Schwarzschild and Kerr solutions is unaffected by the background contribution.

There are other good reasons why one should be interested in working with the new Lagrangian (3.1) rather than with the usual Hilbert Lagrangian. In fact, the superpotential $\boldsymbol{U}(\boldsymbol{\mathcal{L}}_G, \Xi)$ reproduces the usual ADM mass [36] for asymptotically flat space-times [37]. Moreover, in the Reissner-Nordström case it recovers Penrose's quasi-local mass [38] (cf. [39]): when the global Lagrangian is considered (i.e. when also the electrostatic contribution is taken into account), the result we get is—in our opinion—even more convincing from a physical point of view [37].
Actually, in the Schwarzschild and Reissner-Nordström cases, the mass scalar deriving from $\boldsymbol{U}(\boldsymbol{\mathcal{L}}_G, \Xi)$ coincides with a well-known definition of mass for spherically symmetric space-times (see [40] and references quoted therein). This coincidence is limited to a very restricted subclass of solutions, although, e.g., both methods consistently yield the same result for the total mass of a closed FRW universe, i.e. zero [41].

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